Combined Segregated Finite Element method and SUPG for Conjugate Heat Transfer Problems

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Abstract

This paper presents a combined segregated finite element method and Streamline Upwind Petrov-Galerkin method (SUPG) for solving conjugate heat transfer problems where heat conduction in a solid is coupled with heat convection in viscous fluid flow. The Streamline Upwind Petrov-Galerkin method is used for the analysis of viscous thermal flow in the fluid region, while the analysis of heat conduction in solid region is performed by the Galerkin method. The method uses the three-node triangular element with equal-order interpolation functions for all variables of the velocity components, the pressure and the temperature. The main advantage of the presented method is to consistently couple heat transfer along the fluid-solid interface. Two test cases, which are conjugate counter flow heat exchanger and conjugate natural convection and conduction from heated cylinder in square cavity, are selected to evaluate efficiency of the presented method.

1. Introduction

Conjugate heat transfer problems are encountered in many practical applications, where heat conduction in a solid region is closely coupled with heat convection in an adjacent fluid. There are many engineering problems where conjugate heat transfer should be considered such as design of air-cooled packaging, heat transfer enhancement by the finned surfaces, design of thermal insulation, nuclear reactor design, design of solar equipment, heat transfer in a cavity with thermally conducting wall or internal baffle, etc. Most of the studies in this research area, however, employ the finite difference and the finite volume methods as the numerical tools. He et al. [1] studied the conjugate problem using an iterative FDM/BEM method for parallel plate channel with constant outside temperature.

Sugavanam et al. [2] studied a numerical investigation of conjugate heat transfer from a flush heat source on a conductive board in laminar channel flow. Chen and Han [3] presented the solution of a conjugate heat transfer problem using a finite difference SIMPLE-like algorithm. Schäfer and Teschauer [4] used the finite volume method for analysis of both the fluid flow behavior and the solid heat transfer with thermal effect. Kang-Youl Bae et al. [5] studied on natural convection in a rectangular enclosure by using the finite volume method. The results from these researches show that both the finite difference and the finite volume methods can perform very well on the problems of interest, but some assumptions on heat transfer coefficients have to be made in order to compute the temperatures along the fluid-solid interface. Furthermore, deter-mination of the unknown temperatures and the heat fluxes at the fluid-solid interface is normally performed in an iterative way, usually through the use of the artificial heat transfer coefficient.

At present, there are very few publications for solving the conjugate heat transfer problems the finite element method have been proposed in the literature. Misra and Sarkar [6] used the standard Galerkin formulation to solve the continuity, momentum and energy equations simultaneously.

In this paper, the streamline upwind Petrov-Galerkin method [7-8] is selected for the analysis of conjugate heat transfer problems. The method uses triangular elements with equal-order interpolation functions for the velocity components, the pressure and the temperature. A segregated solution algorithm [9-11] is also incorporated to solve the unknown variables separately for improving the computational efficiency. The main advantages of the presented scheme are illustrated and explained by using Figs. 1-2.

Figure 1 shows typical control volumes of the fluid and solid cells along the fluid-solid interface used by the finite volume method. In the figure, the control volumes 1 and 2 are in the fluid region while the control volumes 3 and 4 are in the solid region.
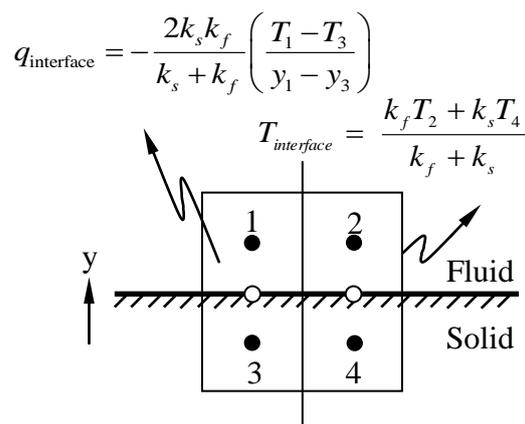
$$q_{\text{interface}} = -\frac{2k_s k_f}{k_s + k_f}\left(\frac{T_1 - T_3}{y_1 - y_3}\right)$$

$$T_{interface} = \frac{k_f T_2 + k_s T_4}{k_f + k_s}$$

Fig. 1. Control volumes across fluid-solid interface used by the finite volume method.

$T_{interface}$ can be

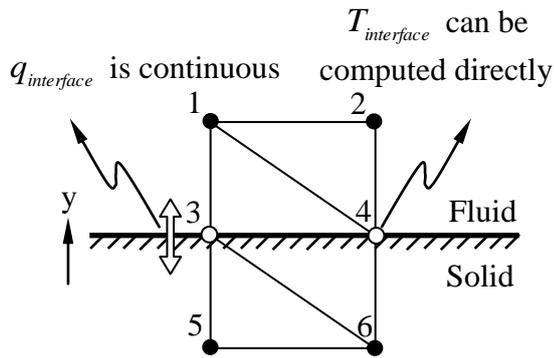$q_{interface}$ is continuous computed directly

Fig. 2. Elements across fluid-solid interface in finite element method.

Because the heat conduction coefficients in solid and fluid regions are different, the harmonic mean of the heat conduction coefficient along the fluid-solid interface was introduced and assumed in the form [12],

$$k_{\text{interface}} = \frac{2k_s k_f}{k_s + k_f} \qquad (1)$$

where k_s and k_f are the heat conduction coefficients in the solid and the fluid region, respectively. The heat flux across the fluid-solid interface was then calculated using the assumed heat conduction coefficient. For the finite element method presented in the paper, the elements along the interface are shown in Fig. 2. The use of the finite element method, for both fluid and solid regions with common nodes along the fluid-solid interface, that provide convenience in analysis computation. At the same time, the use of the single finite element method in both the regions allows the fluid-solid interface temperatures to be computed directly without assuming the heat transfer coefficient. In addition, the continuity of the heat fluxes across the fluid and solid regions along the interface is also preserved automatically.

This paper starts from briefly describing the set of the partial differential equations that satisfy the law of conservation of mass, momentums and energy. Corresponding finite element equations are derived and the element matrices are presented. The computational procedure used in the development of the computer program is then described. Finally, the finite element formulation and the computer program are then verified by solving several examples that have exact solution and numerical solutions from other methods.

2. Theoretical formulation and solution procedure

2.1 Governing equations

The governing equations for conjugate heat transfer problems consist of the conservation of mass which is called the continuity equation, the conservation of momentum in x and y directions, and the conservation of energy, as follows,

Continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2a)$$

Momentum equations,

$$\rho \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \qquad (2b)$$

$$\rho \left[u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right] = -\frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right]$$
$$- \rho g \left(1 - \beta \left(T - T_o \right) \right) \qquad (2c)$$

Energy equation for fluid,

$$\rho c \left[u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right] + \rho Q$$
$$(2d)$$

Energy equation for solid,

$$k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right] = \tilde{Q} \qquad (2e)$$

where u and v are the velocity components in the x and y direction, respectively, ρ is the density, p is the pressure, μ is the viscosity, g is the gravitational acceleration constant, β is the volumetric coefficient of thermal expansion, T is the temperature, T_0 is the reference temperature for which buoyant force in the y-direction vanishes, c is the specific heat, k is the coefficient of thermal conductivity and Q is the internal heat generation rate per unit volume. Equation (2d) can also be used for solving conduction heat transfer in solid by

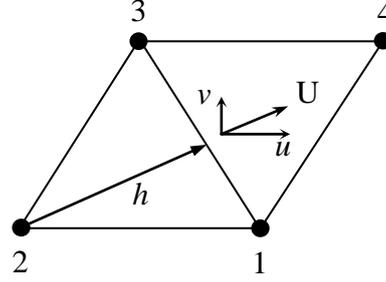setting both velocity components, u and v, as zero.

Fig. 3. The two-dimensional, element size h and streamline directions.

2.2 Finite element formulation

2.2.1 Streamline Upwind Petrov-Galerkin method

The basic idea of the streamline upwind method is to add diffusion, which acts only in the flow direction. Extended to a Petrov-Galerkin formulation, the standard Galerkin weighting functions are modified by adding a streamline upwind perturbation, p, for suppressing the non-physical spatial oscillation in the numerical solution, which again acts only in the flow direction. In this paper, the modified weighting function, W_i, can be expressed as, [8]

$$W_i = N_i + p \qquad (3)$$

$$W_i = N_i + \frac{\alpha h}{2|U|} \left[u \frac{\partial N_i}{\partial x} + v \frac{\partial N_i}{\partial y} \right] \qquad (4)$$

where α is calculated for each element from,

$$\alpha = \alpha_{opt} = \coth Pe - \frac{1}{Pe} \qquad (4a)$$

with $Pe = \frac{|U| h}{2 \tilde{k}}$ and $|U| = \sqrt{u^2 + v^2}$ $\qquad (4b)$

where $\mathbf{Pe}$ is the Peclet numbers, $|U|$ is mean resultant velocity and h is element size as shown in Fig. 3.

2.2.2 Discretization of momentum and energy equations

The three-nodes triangular element is used in this study. The element assumes linear interpolation functions for the velocity components, the pressure, and the temperature as

$$\phi(x, y) = \sum_{i=1}^{3} N_i(x, y)\phi_i \qquad (5)$$

where ϕ is transport property (u, v, p and T) and N_i are the element interpolation functions.

To derive the momentum and the energy equations that correspond to the Streamline Upwind Finite Element scheme and the Streamline Upwind Petrov-Galerkin scheme, the Galerkin method of weighted residuals is employed by multiplying Eqs. (2b-d) with the weighting function, N_i, except for the convection terms which the special treatment as described in the above sections is used. Integration by parts are then performed using the Gauss theorem to yield the element equations in the form,

Momentum equations,

$$[A]\{u\} = \{R_{px}\} + \{R_u\} \qquad (6a)$$

$$[A]\{v\} = \{R_{py}\} + \{R_v\} + \{R_{gy}\} \qquad (6b)$$

Energy equation,

$$\left[A^T\right]\{T\} = \{R^T\} + \{R_Q^T\} \qquad (7)$$

where the coefficient matrices $[A]$ and $\left[A^T\right]$ contain the known contributions from the convection and diffusion terms. Details of these matrices can be found in ref [10].

2.2.3 Discretization of pressure equation

To derive the pressure equation, the method of weighted residuals is applied to the continuity equation, Eq. (2a). Because the pressure term does not appear in the continuity equation, the relation between velocity components and pressure are thus required. Such relations can be derived from the momentum equations, Eqs. (6a-b) as,

$$A_{ii}u_i = -\sum_{j \neq i} A_{ij}u_j + f_i^u - \int_{\Omega} N_i \frac{\partial p}{\partial x} d\Omega \qquad (8a)$$

$$A_{ii}v_i = -\sum_{j \neq i} A_{ij}v_j + f_i^v - \int_{\Omega} N_i \frac{\partial p}{\partial y} d\Omega \qquad (8b)$$

where f_i^u and f_i^v are the surface integral terms and the source term due to the buoyancy.

By applying Eqs. (8a-b) into the continuity equations, the pressure equations can be written in matrix form with unknowns of the nodal pressure as

$$[K]\{p\} = \{F_u\} + \{F_v\} + \{F_b\} \qquad (9)$$

where the details for these element matrices can also found in Ref. [10].

The above element equations are assembled to yield the global equations for the velocity components, the temperature and the pressure equations. Appropriated boundary conditions are then applied prior to solving for the new velocity components, temperature and pressure values.

2.2.5 Computational procedure

The computational procedure starts from assuming initial nodal velocity components, pressures, and temperatures. The new nodal temperatures are computed using Eq. (7). The new nodal velocity components and pressures are then computed using Eqs. (6a-b) and (9), respectively. The nodal velocity components are then updated using Eqs. (8a-b) with the computed nodal pressures. This process is continued until the specified convergence criterion is met. Such segregated solution procedure helps reducing the computer storage because the equations for the velocity components, the pressure, and the temperature are solved separately.

3. Results

In this section, two example problems are presented. The first example, Conjugate counter flow heat exchanger, is chosen to evaluate the finite element formulations and to validate the developed computer programs. The second example, conjugate natural convection and conduction from heated cylinder in square cavity, are used to illustrate the capability of the presented schemes in the analysis of conjugate heat transfer problems.

3.1 Conjugate counter flow heat exchanger

The first example for validating the numerical scheme, a conjugate counter flow heat exchanger problem is selected as the second test case. This heat exchanger consists of two parallel flow passages with widths a_1 and a_3, separated by a solid plate with thickness of a_2 as shown in Fig. 4. The outer walls of the flow passages are assumed to be adiabatic. The same properties and uniform inlet velocity and temperature profiles are assumed for the hot and cold fluids.
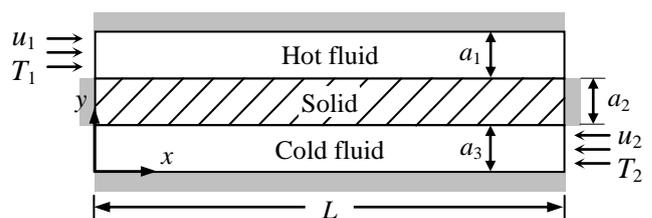

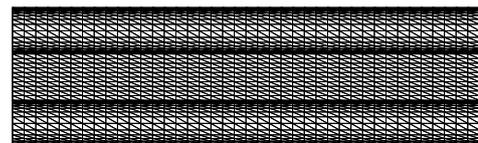Fig. 4. A conjugate counter flow heat exchanger.

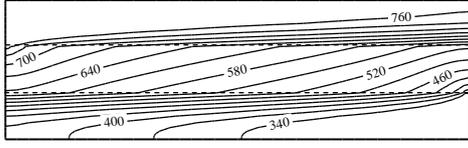Fig. 5. Finite element model for conjugate counter flow heat exchanger.

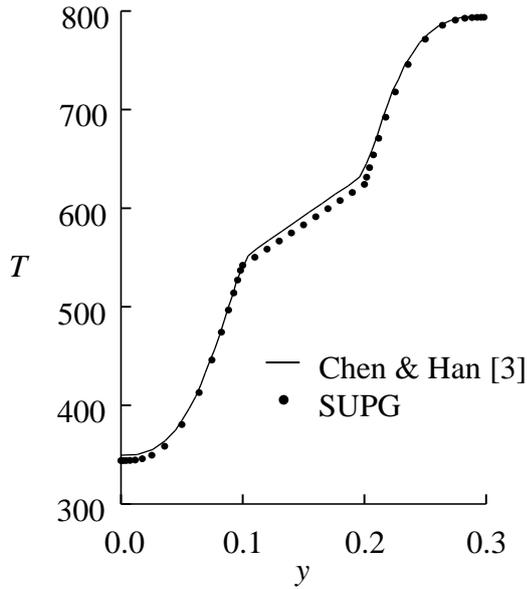Fig. 6. Predicted temperature contours at $K = 5$ for a conjugate counter flow heat exchanger.

Fig. 7. The temperature profiles at $x = L/2$ at $K = 5$ for a conjugate counter flow heat exchanger.

The parameters adopted in the computation are as follows, geometrical sizes $a_1 = a_2 = a_3 = 0.1$ and $L = 1.0$, the flow parameters in upper channel $u_1 = 0.2$, $T_1 = 800$, $\mathrm{Re} = 133.33$ and $\mathrm{Pr} = 0.75$, the flow parameters in lower channel $u_2 = 0.1$, $T_2 = 300$, $\mathrm{Re} = 66.67$ and $\mathrm{Pr} = 0.75$, conduction ratio $K = 5$. The finite element model, consisting of 1,763 nodes and 3,360 triangles as shown in Fig. 5, is used in this study.

Fig. 6 shows the predicted temperature contours in entire domain. The predicted temperature distributions at $x = L/2$ from presented scheme is compared with the finite volume results from Chen and Han [3] as shown in Fig. 7. The figure also shows good agreement of the solutions.

3.2 Conjugate natural convection and conduction from heated cylinder in square cavity

The last example of a high temperature cylinder enclosed by a square cavity as shown in Fig. 8, is selected to demonstrate the use of the presented method for the problem with a more complex geometry. Both the vertical side walls of the square cavity are isothermal. The upper horizontal boundary is surrounded by a solid material. The upper boundary of this solid region is considered as adiabatic. The lower horizontal boundary of the fluid cavity is also an adiabatic boundary. Due to the symmetry of flow solution, only the right half of the enclosure was analyzed. The finite element model consisting of 1,821 nodes and 3,450 triangles, as shown in Fig. 9, is used in this study. Figure 10 shows the predicted streamline and temperature contours vary with the thermal conductivity ratios and the Rayleigh numbers. Figure 11 shows the predicted temperature distributions of lower adiabatic boundary at $y = 0$, interface of fluid-solid regions at $y = 1$,

and the upper adiabatic boundary at $y = 1.2$. This picture represents the different thermal conductivity ratios of $K =$ 0.1, 1, 5 and 10, respectively, at the Rayleigh number of 10^4 and is compared with the results from Dong and Li [13]. The figure shows good agreement of the solutions obtained from the presented scheme.

In addition, the average Nusselt number, $\overline{Nu}$, was also investigated in this research, the average Nusselt number can be calculated by,

$$Nu_{mean} = \frac{1}{2\pi r} \oint Nu \ ds \qquad (20)$$

which

$$Nu = \left| \frac{\partial T}{\partial x} \right| \times \frac{|x - x_c|}{r} + \left| \frac{\partial T}{\partial y} \right| \times \frac{|y - y_c|}{r} \qquad (21)$$

where x_c, y_c are the center coordinates of the high temperature cylinder and r is radius of cylinder.
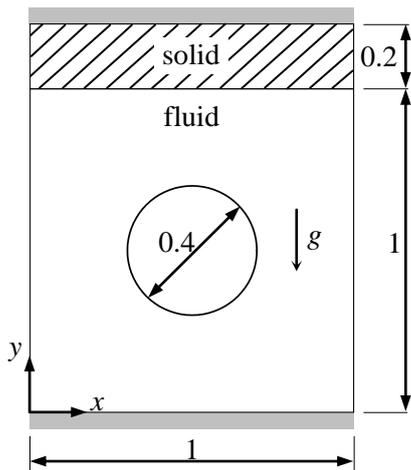

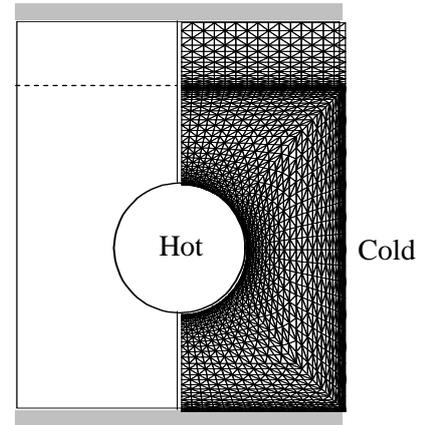Fig. 8. Conjugate natural convection from heated cylinder in square cavity problem.

Fig. 9. Finite element model for heated cylinder in square cavity problem.

Table 1 gives a comparison of $\overline{Nu}$ at the cylinder's surface with results of Dong and Li. From the table, it can be concluded that the overall mean Nusselt number increases with the increase of thermal conductivity ratios and the Rayleigh numbers. The table shows good agreement of the solutions obtained from the two methods.

4. Conclusions

A coupled finite element method for conjugate heat transfer problems was presented. The method combines the viscous thermal flow analysis of the fluid region and the heat transfer analysis in the solid region together.
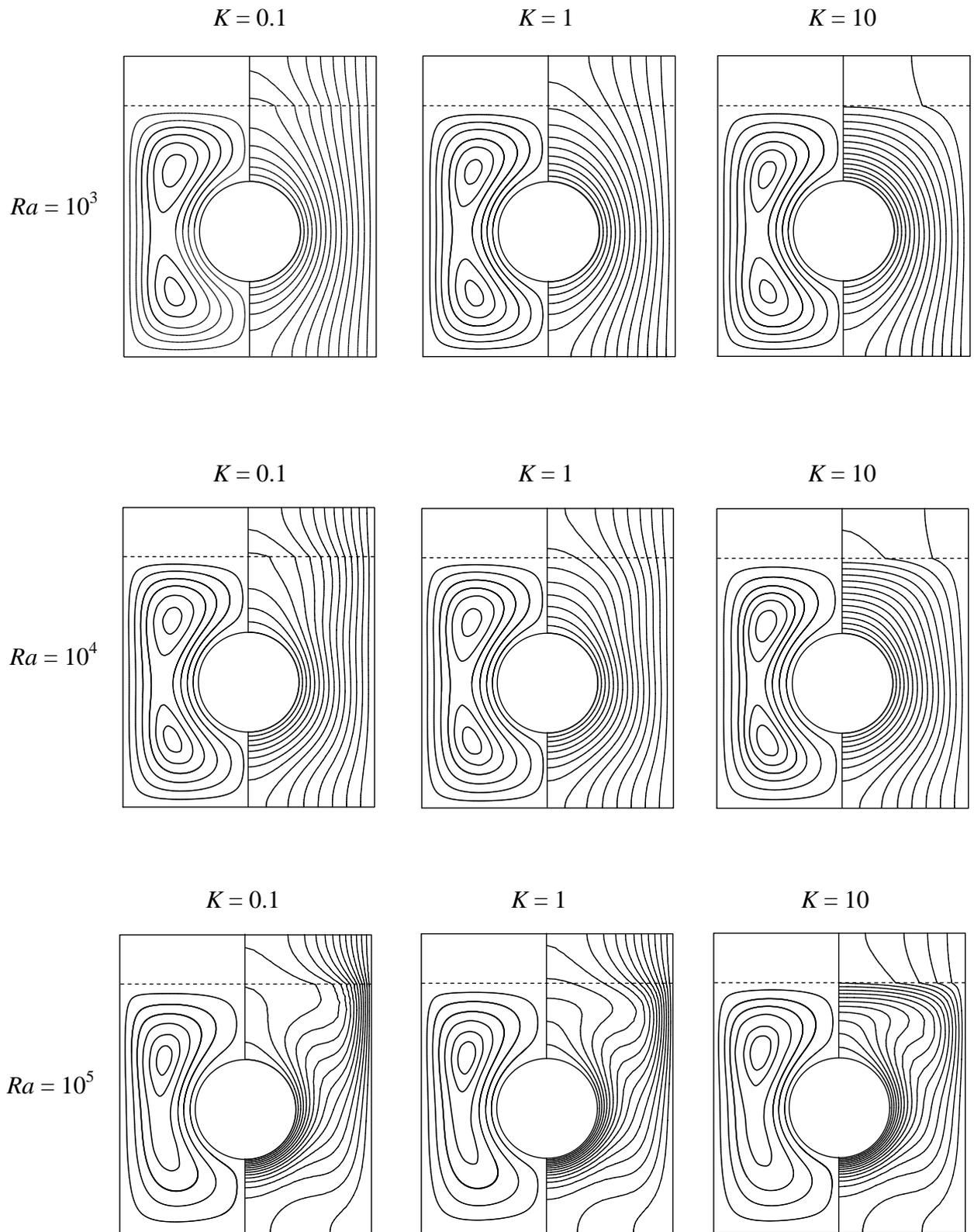
$K = 0.1$ $K = 1$ $K = 10$

$Ra = 10^3$

$K = 0.1$ $K = 1$ $K = 10$

$Ra = 10^4$

$K = 0.1$ $K = 1$ $K = 10$

$Ra = 10^5$

Fig. 10. Streamline and temperature contours for $K = 0.1$, 1 and 10, at $Ra = 10^3$, 10^4, and 10^5.

K = 0.1

K = 10.0
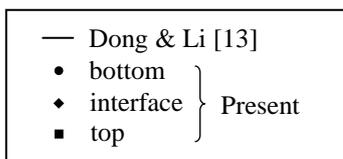
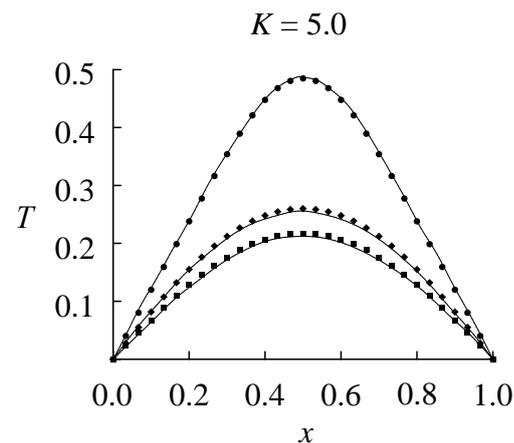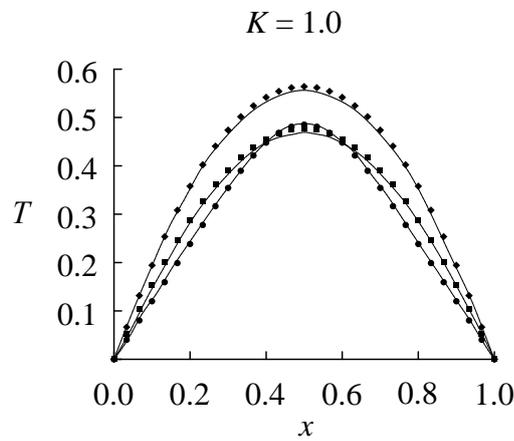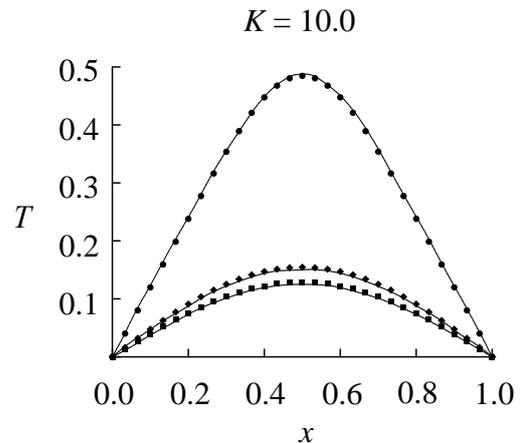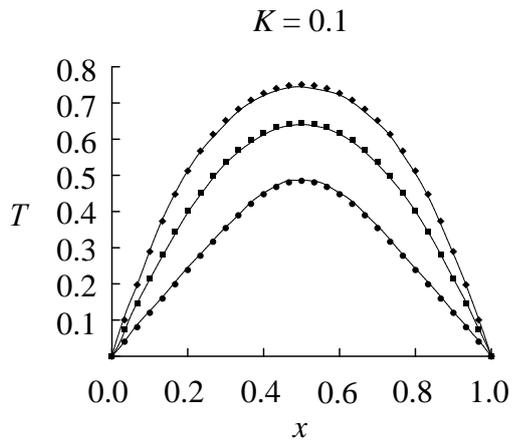Fig. 11. Compared the temperature distributions for conjugate natural convection and conduction from heated cylinder in square cavity for K = 0.1, 1, 5 and 10, all at Ra = 10^4.

K = 1.0

K = 5.0

— Dong & Li [13]
● bottom
◆ interface } Present
■ top

The finite element formulation and its computational procedure were first described. The flow analysis used a segregated solution algorithm to compute the velocities, the pressure and the temperature separately for improving the computational efficiency. The convection terms in the momentum and the energy equations were treated by the Streamline Upwind Petrov-Galerkin method to suppress the non-physical spatial oscillation in the numerical solutions. All the finite element equations were derived and a corresponding computer program was developed. The efficiency of the coupled finite element method has been evaluated by two examples that were previously performed using other methods.

Table 1. Variation of the overall mean Nusselt numbers along cylinder (%Δ from Ref. [13])

Ra		Conductivity ratio, $K = k_s / k_f$			
		0.1	1	5	10
10^3	Present	3.78	3.95	4.19	4.28
10^4	Dong & Li [13]	4.05	4.20	-	4.49
10^4	Present	3.99 (1.48%)	4.13 (1.67%)	4.35	4.43 (1.34%)

These examples highlight the benefit of the combined finite element method that can simultaneously model and solve both the fluid and solid regions, as well as to compute the temperatures along the fluid-solid interface directly.

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