Characteristic-Based Split Finite Element Algorithm for Viscous Incompressible Flow Problems

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Abstract

A finite element method for solving the steady viscous incompressible flow problems using the characteristic-based split algorithm with equal-order three nodes triangular element is presented. The performance of the method has been evaluated by solving several problems that have exact and numerical solutions. An adaptive meshing technique is incorporated to improve the unsteady solution accuracy. The performance of the combined adaptive mesh movement technique and the characteristic-based split algorithm is illustrated by solving the problem of the flow past a cylinder.

Keywords: Characteristic-based split algorithm, Finite element method, Viscous Incompressible flow.

1. Introduction

The use of the finite element method to solve fluid flow problems has increased its popularity during the past decades. This is due to the fact that the governing differential equations for general flow problems consist of several coupled equations which inherently nonlinear of convective terms. Together with complex flow geometries and general boundary conditions that increase the difficulty of the problem, the finite element method is more attractive as compared to other numerical methods. During the past twenty years, several finite element algorithms [1-3] were developed to alleviate the computational effort in order to capture complex flow fields and to suppress the spatial oscillations from the weakness of the standard Galerkin method on the convective terms discretization.

In this paper, the characteristic-based split algorithm or the "CBS algorithm" is combined with adaptive meshing technique to solve viscous incompressible flow problems. The characteristic-based split algorithm is selected for the flow analysis because of its capability to provide the solution accuracy for both steady and unsteady of fluid dynamics problems. The algorithm allows equal-order interpolation functions to be used for all variables, therefore the complexity in deriving the finite element equations is reduced. An adaptive meshing technique is applied to improve the finite element solution accuracy by placing small elements in the regions of large change in the solution gradients to increase the solution accuracy. The predicted solutions are compared with the exact solution, and the prior numerical result. Finally, the adaptive remeshing technique is combined with the CBS algorithm to analyze the flow behavior past a cylinder.

2. Finite Element Formulations

The fundamental laws used to solve fluid motion in a general form are the law of conservation of mass, conservation of momentum, which constitute a set of coupled, nonlinear, partial differential equations. These governing differential equations for the two-dimensional isothermal laminar viscous incompressible flow are,

Mass Conservation:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial x_i} = 0 \qquad (1a)$$

Momentum Conservation:

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}-\frac{1}{\rho}\frac{\partial p}{\partial x_i} \quad (1b)$$

where i, j is the tensor index referring to the x and y axis, respectively, ρ is the fluid density, u_i are the velocity components, τ_{ij} are the stress components, and p is the fluid pressure.

The characteristic-based split algorithm was first introduced by Zienkiewicz and Codina [4] for solving different categories of fluid dynamics problems. The procedure of characteristic-based split algorithm is to use the characteristic-Galerkin method and the operator splitting in order to solve uncoupled equations for temporal discretization, while the method of weighted residuals with Galerkin's criteria is used for spatial discretization to derive the finite element equations.

2.1 Temporal Discretization

The characteristic-based split algorithm for viscous incompressible flow analysis consists of three steps. In the first step, the intermediate velocities of the momentum equations are calculated by omitting the pressure gradient terms. In the second step, the change of the intermediate velocities is used to determine the pressure by solving continuity equation. Finally, the pressure is used as a corrector to update the velocities of the momentum equations in the last step. These three steps can be written in the semi-implicit form as follows,

Step 1: The intermediate velocity equations,

$$\Delta u_i^* = \Delta t\left[-u_j\frac{\partial u_i}{\partial x_j}+\frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}\right.$$
$$\left.+\frac{\Delta t}{2}u_k\frac{\partial}{\partial x_k}\left(u_j\frac{\partial u_i}{\partial x_j}+\frac{1}{2\rho}\frac{\partial p}{\partial x_i}\right)\right]^n \quad (2)$$

Step 2: The continuity equation,

$$\frac{\partial^2 p^{n+1}}{\partial x_i \partial x_i}=\frac{\rho}{\Delta t}\left[4\frac{\partial u_i^{\,n}}{\partial x_i}+2\frac{\partial \Delta u_i^*}{\partial x_i}\right]-\frac{\partial^2 p^n}{\partial x_i \partial x_i} \quad (3)$$

Step 3: The velocity correction equations,

$$\Delta u_i = u_i^{\,n+1}-u_i^{\,n}=\Delta u_i^* - \Delta t\frac{1}{2\rho}\left[\frac{\partial p^{n+1}}{\partial x_i}+\frac{\partial p^n}{\partial x_i}\right] \quad (4)$$

2.2 Spatial Discretization

The three-node triangular element is employed in this study. Both velocity and pressure variables discretization use standard Galerkin with the assume linear interpolation shape functions as,

$$u = N_\alpha(x,y)u_\alpha \quad (5a)$$
$$v = N_\alpha(x,y)v_\alpha \quad (5b)$$
$$p = N_\alpha(x,y)p_\alpha \quad (5c)$$

where $\alpha = 1,2,3$ and N are the element interpolation functions.

The method of weighted residuals with Galerkin's criteria is employed to discretize the finite element equations by multiplying Eqs. (2)-(4) with the weighting functions and performing integration by parts using the Gauss theorem [5] to yield the element equations shown in the steps below,

Step 1: The intermediate velocity equations,

$$[M]\{\Delta u_i^*\} = -\Delta t\left[\{C_u\}+\{K_u\}-\{R_{ku}\}\right]^n$$
$$-\frac{\Delta t^2}{2}\left[\{C_s\}+\{P_s\}-\{R_{cs}\}-\{R_{ps}\}\right]^n \quad (6)$$

Step 2: The continuity equation,

$$[K_p]\{p\}^{n+1}=\frac{4}{\Delta t}\{C_p\}^n-\{P_p\}^n \quad (7)$$

Step 3: The continuity equation,

$$[M]\{\Delta u_i\}^{n+1}=[M]\{\Delta u_i^*\}-\frac{\Delta t}{2}\{P_u\} \quad (8)$$

In the above equations, the element matrices written in the integral form are,

$$[M]=\int_\Omega \{N\}\lfloor N\rfloor\, d\Omega \quad (9a)$$

$$\{C_p\}=\int_\Omega \{N\}\left(u_j\frac{\partial u_i}{\partial x_j}\right)d\Omega \quad (9b)$$

$$\{K_u\}=\frac{1}{\rho}\int_\Omega\left\{\frac{\partial N}{\partial x_j}\right\}\tau_{ij}\,d\Omega \quad (9c)$$

$$\{R_{ku}\}=\frac{1}{\rho}\int_\Gamma \{N\}\tau_{ij}\,\hat{n}_j\,d\Gamma \quad (9d)$$

$$\{C_s\}=\bar{u}_k\int_\Omega\left\{\frac{\partial N}{\partial x_k}\right\}\left(u_j\frac{\partial u_i}{\partial x_j}\right)d\Omega \quad (9e)$$

$$\{R_{cs}\} = \bar{u}_k \int_\Gamma \{N\} \left(u_j \frac{\partial u_i}{\partial x_j} \right) \hat{n}_k \, d\Gamma \qquad (9f)$$

$$\{P_s\} = \frac{1}{2\rho} \bar{u}_k \int_\Omega \left\{ \frac{\partial N}{\partial x_k} \right\} \left(\frac{\partial p}{\partial x_i} \right) d\Omega \qquad (9g)$$

$$\{R_{ps}\} = \frac{1}{2\rho} \bar{u}_k \int_\Omega \{N\} \left(\frac{\partial p}{\partial x_i} \right) \hat{n}_k \, d\Omega \qquad (9h)$$

$$\{K_p\} = \int_\Omega \left\{ \frac{\partial N}{\partial x_i} \right\} \left\lfloor \frac{\partial N}{\partial x_i} \right\rfloor d\Omega \qquad (9i)$$

$$\{C_p\} = \rho \int_\Omega \left\{ \frac{\partial N}{\partial x_i} \right\} \left(u_i + \frac{1}{2} \Delta u_i^* \right) d\Omega \qquad (9j)$$

$$\{P_p\} = \int_\Omega \left\{ \frac{\partial N}{\partial x_i} \right\} \frac{\partial p}{\partial x_i} d\Omega \qquad (9k)$$

$$\{P_u\} = \frac{1}{\rho} \int_\Omega \{N\} \left(\frac{\partial p^n}{\partial x_i} + \frac{\partial p^{n+1}}{\partial x_i} \right) d\Omega \qquad (9l)$$

where $\bar{u}$ is the average velocity of the element. The semi-implicit form of CBS algorithm is conditionally stable. The permissible time step is governed by,

$$\Delta t = \sigma \frac{h^2}{2\upsilon} \qquad (10)$$

where σ is the Courant number ($0<\sigma \leq 1$) and υ is the kinematic viscosity.

3. Results

In this section, three example problems are presented. The first and second examples, the unsteady flow over a moving boundary and the lid-driven cavity flow, are chosen to evaluate the algorithm performance. The third examples, the flow past a cylinder, is used to illustrate the capability of the combined adaptive meshing technique and the characteristic-based split algorithm for viscous incompressible flow problems.

3.1 The Unsteady Flow over a Moving Boundary

The first example used for evaluating the CBS algorithm is the unsteady flow over a moving boundary. This example is selected because it has exact solutions for comparison. The problem statement is described in Fig. 1. The initial and boundary conditions of problem are given by $u(0,t) = 1.0$ for $t > 0$ and $u(y,0) = u(1,t) = 0.0$. Figure 2 shows the comparison of the numerical results with the exact solution at various times [6]. The figure shows good agreement of the solutions.

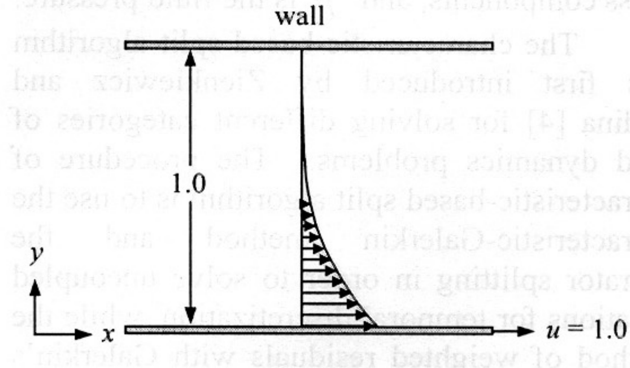

Fig. 1. Problem statement of the unsteady flow over moving boundary

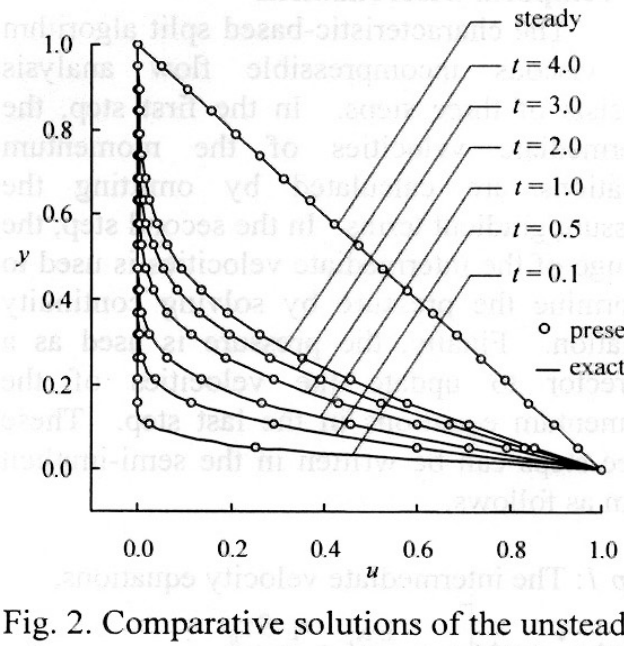

Fig. 2. Comparative solutions of the unsteady flow over moving boundary with $Re = 100$

3.2 The Lid-Driven Cavity Flow

The problem of the flow circulation in a closed cavity driven by a moving lid has been widely used to evaluate new method formulation. The flow circulation in a unit square cavity is induced by a moving lid at the velocity of u =1.0 to the right. The problem statement and the finite element model consisting of 10,201 nodes and 20,000 elements are illustrated in Fig. 3. Figures 4(a)-(d) show the predicted streamline contours at various times. Figure 5 shows good agreement of the velocity profiles along the cavity centered lines obtained from the present algorithm and those presented Ref. [7].

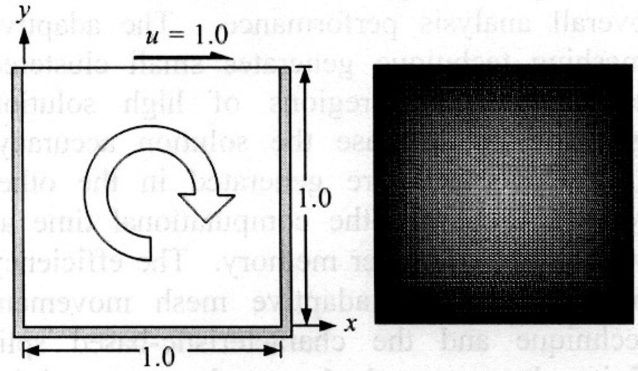

Fig. 3. Problem statement and finite element model of the lid-driven cavity flow problem

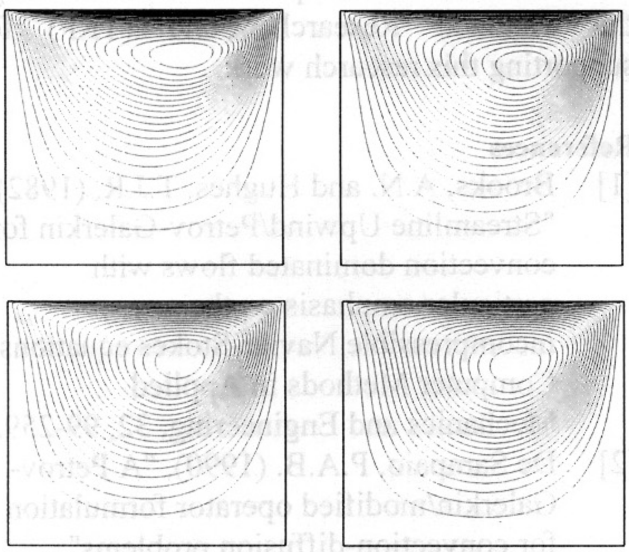

Fig. 4. Predicted streamline contours at time: (a) $t = 1$; (b) $t = 2$; (c) $t = 4$; and (d) steady

3.3 The Flow Past a Cylinder

To evaluate the performance of the adaptive meshing technique [8] combining with the characteristic-based split algorithm for viscous incompressible analysis, the problem of flow past a cylinder is selected. The flow past a cylinder is a fundamental fluid mechanics problem of practical importance.

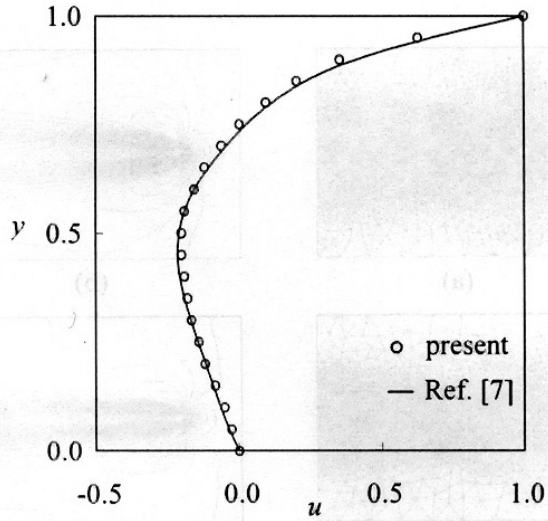

Fig. 5. Predicted steady state streamline contours

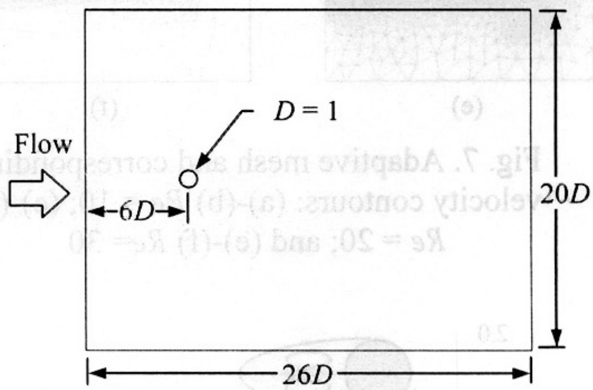

Fig. 6. Problem statement of the flow past cylinder

The flow field over the cylinder is symmetric at low values of Reynolds number. As the Reynolds number increases, flow begins to separate behind the cylinder causing vortex shedding which is an unsteady phenomenon. The computational domain is shown in Fig. 6. The problem is analyzed with different values of the Reynolds numbers at 10, 20 and 30. Figure 7 shows the time evolution of the adaptive mesh and velocity contours at three Reynolds numbers. Figure 8 compares the reattachment length as the function of Reynolds numbers with previous results [9]. The figure shows good agreement of the solutions.

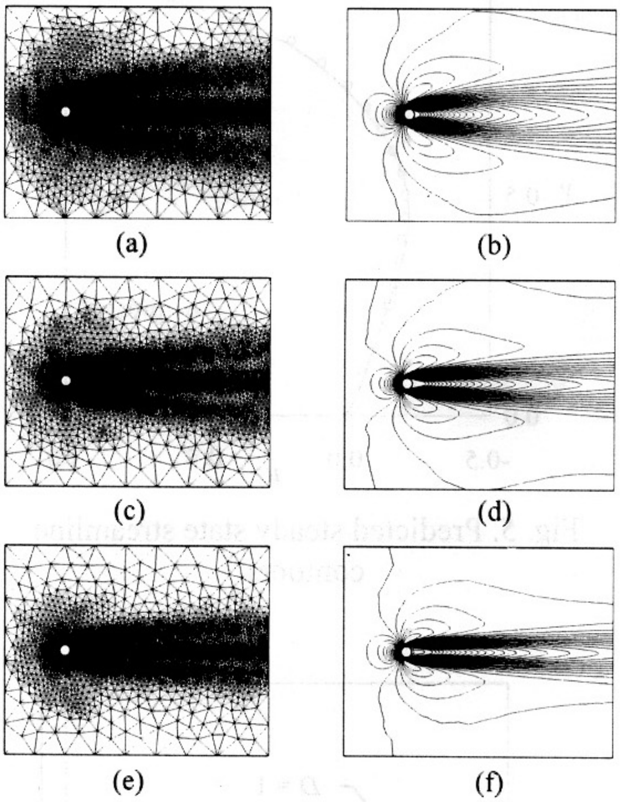

Fig. 7. Adaptive mesh and corresponding velocity contours: (a)-(b) Re = 10; (c)-(d) Re = 20; and (e)-(f) Re= 30

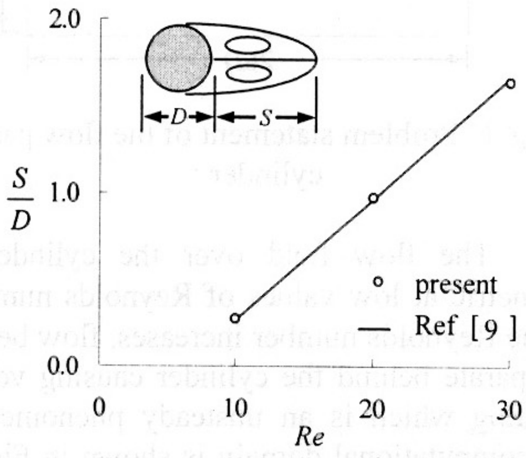

Fig. 8. Comparative of reattachment length

4. Conclusions

The combined adaptive mesh movement technique and a finite element algorithm for steady and unsteady viscous incompressible flow analyses was presented. The finite element equations were derived from the governing Navier-Stokes differential equations using the characteristic-based split algorithm. All finite element matrices were derived in closed-form and a corresponding computer program was developed. Two examples with exact and numerical solutions were used to validate the performance of the characteristic-based split algorithm. The method was also combined with an adaptive meshing technique to further increase the overall analysis performance. The adaptive meshing technique generates small clustered elements in the regions of high solution gradients to increase the solution accuracy. Larger elements are generated in the other regions to reduce the computational time as well as the computer memory. The efficiency of the combined adaptive mesh movement technique and the characteristic-based split finite element method was demonstrated by using the example of a flow past a cylinder.

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